REGULARITY OF ISOPERIMETRIC SETS IN $\mathbb{R}^2$ WITH DENSITY

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ABSTRACT. We consider the isoperimetric problem in $\mathbb{R}^n$ with density for the planar case $n = 2$. We show that, if the density is $C^{0,\alpha}$, then the boundary of any isoperimetric is of class $C^{1,\frac{\alpha}{3-2\alpha}}$. This improves the previously known regularity.

1. Introduction

The isoperimetric problem in $\mathbb{R}^n$ with density is a classical problem, which has received much attention in the last decade. The idea is quite simple: given a l.s.c. function $f : \mathbb{R}^n \to (0, +\infty)$, usually called "density", for any set $E \subseteq \mathbb{R}^n$ we define the volume $V_f(E)$ and the perimeter $P_f(E)$ as

$$V_f(E) := \int_E f(x)\, dx\,, \qquad\qquad P_f(E) := \int_{\partial^* E} f(x)\, d\mathscr{H}^{n-1}(x)\,,$$

where the subscript reminds the fact that volume and perimeter are done with respect to f, and where $\partial^* E$ is the reduced boundary of E (to read this paper there is no need to know what the reduced boundary is, since under our assumptions this will always coincide with the usual topological boundary ∂E). The literature on this problem is huge, a short and incomplete list is [2, 4, 7, 20, 8, 10, 13, 16, 22, 23]).

The main questions about the isoperimetric problem with density are of course the existence and regularity of isoperimetric sets; concerning the existence, now a lot is known (see for instance [22, 12, 9]). Since in this paper we are dealing with the regularity, let us briefly recall the main known results. The first, very important one, can be found in [19, Proposition 3.5 and Corollary 3.8] (see also [1, 3]).

Theorem 1.1. *Let f be a $C^{k,\alpha}$ density on $\mathbb{R}^n$, with $k \geq 1$. Then the boundary of any isoperimetric set is $C^{k+1,\alpha}$, except for a singular set of Hausdorff dimension at most $n-8$. If f is just Lipschitz and $n = 2$, then the boundary is of class $C^{1,1}$.*

It is important to point out that the above result requires at least a $C^{1,\alpha}$ regularity for f (or Lipschitz in the 2-dimensional case). The reason is rather simple: namely, most of the standard techniques to get the regularity make sense only if f is at least Lipschitz (for a discussion on this fact, see [10, Section 5]). In particular, the following observation can be particularly insightful. In order to get regularity of an isoperimetric set E, a standard idea is to build some "competitor" F, which behaves better than E where the boundary of E is not regular enough; however, in order to get some contradiction, we must ensure that F has the same volume as E, since otherwise the isoperimetric property of E cannot be used. On the other hand, it can be complicate to build the set F taking its volume into account, since while defining F one is

interested in its perimeter. As a consequence, it is extremely useful to have the so-called "$\varepsilon - \varepsilon$ property", which roughly speaking says the following: it is always possible to modify the volume of a set F of a small quantity ε, increasing its perimeter of at most $C|\varepsilon|$ for some fixed constant C; if this is the case, one can then "adjust" the volume of F so that it coincides with the one of E. This property was already discussed by Allard, Almgren and Bombieri since the 1970's (see for instance [1, 2, 3, 4, 7]), and it has been widely used in most of the papers about the regularity in this context since then. Unfortunately, while the $\varepsilon - \varepsilon$ property is rather simple to establish when f is at least Lipschitz, it is false if f is not Lipschitz.

To get anyhow some regularity for the isoperimetric sets in case of low regularity of f, in the recent paper [10] we have introduced and proved a weaker property, called the "$\varepsilon - \varepsilon^\beta$ property", which basically says that the volume of a set can be modified of ε, increasing the perimeter of at most $C|\varepsilon|^\beta$. Since we will use this property in the present paper, we give here the result (the actual result proved in [10] is more general, but we prefer to claim here the simpler version that we are going to need).

Theorem 1.2 ([10], Theorem B). *Let $E \subseteq \mathbb{R}^n$ be a set of locally finite perimeter, and f an α-Hölder density for some $0 < \alpha \leq 1$. Then, for every ball B with nonempty intersection with $\partial^* E$, there exist two constants $\bar\varepsilon$, $C > 0$ such that, for every $|\varepsilon| < \bar\varepsilon$, there is a set $\widetilde{E}$ satisfying*

$$\widetilde{E} \Delta E \subset\subset B\,, \qquad V_f(\widetilde{E}) = V_f(E) + \varepsilon\,, \qquad P_f(\widetilde{E}) \leq P_f(E) + C|\varepsilon|^\beta\,, \tag{1.1}$$

where $\beta = \beta(\alpha, n)$ is defined by

$$\beta = \beta(\alpha, n) := \frac{\alpha + (n-1)(1-\alpha)}{\alpha + n(1-\alpha)}\,,$$

so for $n = 2$ it is $\beta = \frac{1}{2-\alpha}$.

By using the classical regularity results (see for instance [5, 24]), and modifying the arguments in order to make use of the $\varepsilon - \varepsilon^\beta$ property instead of the $\varepsilon - \varepsilon$ one (which is false), we then obtained the following regularity result (see [24] for a definition of porosity).

Theorem 1.3 ([10], Theorem 5.7). *Let E be an isoperimetric set in $\mathbb{R}^n$ with a density $f \in \mathrm{C}^{0,\alpha}$, with $0 < \alpha \leq 1$. Then $\partial^* E = \partial E$ is of class $C^{1,\frac{\alpha}{2n(1-\alpha)+2\alpha}}$. In particular, if $n = 2$ then ∂E is $\mathrm{C}^{1,\frac{\alpha}{4-2\alpha}}$. If f is only bounded above and below, then it is still true that $\partial^* E = \partial E$, and moreover E is porous.*

The main result of the present paper is the following stronger regularity result for the bi-dimensional case.

Theorem A (Regularity of isoperimetric sets). *Let $f : \mathbb{R}^2 \to (0, +\infty)$ be a $\mathrm{C}^{0,\alpha}$ density, for some $0 < \alpha \leq 1$. Then, every isoperimetric set E has a boundary of class $\mathrm{C}^{1,\frac{\alpha}{3-2\alpha}}$.*

It is worthy observing that the regularity obtained above is still less than the $\mathrm{C}^{1,\alpha}$ regularity that one could expect just by looking at Theorem 1.1, but it is better than the previously known regularity given by Theorem 1.3. In particular, notice that there is a substantial improvement between $\mathrm{C}^{1,\frac{\alpha}{4-2\alpha}}$ and $\mathrm{C}^{1,\frac{\alpha}{3-2\alpha}}$. Indeed, the second exponent is not just merely bigger than the

first one, there is also a much deeper difference: namely, when α goes to 1, the first exponent goes to $1/2$, while the second goes to 1. In particular, our Theorem A gives also a proof that for f Lipschitz an isoperimetric set is $\mathrm{C}^{1,1}$, as stated in Theorem 1.1.

We conclude this introduction with a couple of remarks. First of all, the fact that the regularity of the "old" Theorem 1.3 does never exceed $\mathrm{C}^{1,\frac{1}{2}}$ is not strange, since this is the best regularity that can be obtained via the classical methods, not really using the (non-local) isoperimetric property of a set E, but only the weaker (local) fact that E is an ω-minimizer of the perimeter. To get anything better than $\mathrm{C}^{1,\frac{1}{2}}$, one has really to use the non-locality of the fact that E is an isoperimetric set, as we do in the present paper by using the non-local $\varepsilon - \varepsilon^\beta$ property. The second remark is about the sharp regularity exponent that one can obtain for an isoperimetric set with a $\mathrm{C}^{0,\alpha}$ density: we do not believe that our exponent of Theorem A is sharp, but we are also not sure whether it is possible to reach the $\mathrm{C}^{1,\alpha}$ regularity, similarly to what happens for $k \geq 1$ in the classical case.

1.1. **Notation.** Let us briefly present the notation of the present paper. The density will always be denoted by $f : \mathbb{R}^2 \to (0, +\infty)$; keep in mind that, since we want to prove Theorem A, the function f will always be at least continuous. For any set $E \subseteq \mathbb{R}^2$, we call $V_f(E)$ and $P_f(E)$ its volume and perimeter. For any $z \in \mathbb{R}^2$ and $\rho > 0$, we call $B_\rho(z)$ the ball centered at z with radius ρ. Given two points $x, y \in \mathbb{R}^2$, we denote by xy, $\ell(xy)$, and $\ell_f(xy)$ the segment connecting the two points, its Euclidean length, and its length with respect to the density f (that is, $\int_{xy} f(t)dt$). Given three points a, b, c, we will denote by $a\widehat{b}c$ the angle between the segments ab and bc. Let now $E \subseteq \mathbb{R}^2$ be an isoperimetric set; then, by Theorem 1.3 we already know that the boundary of E is of class $\mathrm{C}^{1,\frac{\alpha}{4-2\alpha}}$, hence in particular it is locally Lipschitz. As a consequence, for any two points x, y which belong to the same connected component of ∂E and which are very close to each other (with respect to the diameter of this connected component), the shortest curve in ∂E connecting x and y is well-defined. We denote by $\widehat{xy}$ this curve, and again by $\ell(\widehat{xy})$ and $\ell_f(\widehat{xy})$ we denote its Euclidean length, and its length with respect to the density f. The letter C is always used to denote a large constant, which can increase from line to line, while M is a fixed constant, coming from the α-Hölder property.

2. Proof of the main result

This section is devoted to the proof of our main result, Theorem A. Most of the proof consists in studying the situation around few given points, so let us fix some useful particular notation; Figure 1 helps with the names of the points. Let us fix an isoperimetric set E and a point z on ∂E. Let $\rho \ll 1$ be a very small constant, much smaller than the length of the connected component $\gamma \subseteq \partial E$ containing z, and of the distance between z and the other connected components of ∂E (if any). Since we know that γ is a $\mathrm{C}^{1,\frac{\alpha}{4-2\alpha}}$ curve, we can fix arbitrarily an orientation on it; hence, let us define four points x, $\bar{x}$, y, $\bar{y}$ in $\partial B_\rho(z)$ as follows: among the points of γ which belong to $\partial B_\rho(z)$ and which are *before* z, we call $\bar{x}$ the closest one to z, and x the farthest one (in the sense of the paremeterization of γ). We define analogously $\bar{y}$ and y *after* z: of course, $\bar{x}$

and x may coincide, as well as y and $\bar{y}$. Moreover, we call δ the angle $z\widehat{x}y$, and

$$l := \ell(\widehat{x\bar{x}}) + \ell(\widehat{y\bar{y}}) \,.$$

Finally, we introduce the following set F, which we will use as a competitor to E (after adjusting its area).

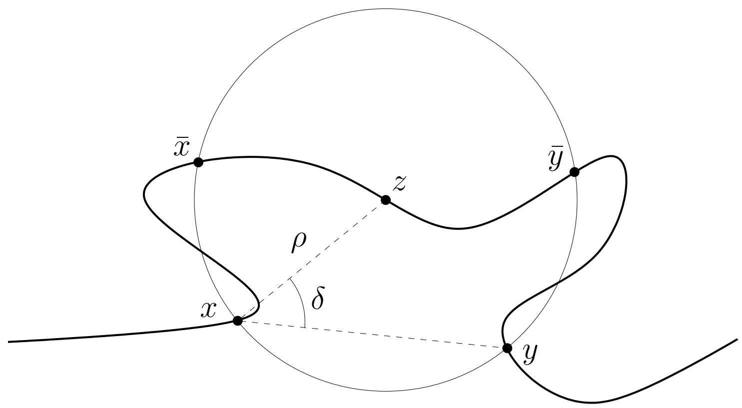

FIGURE 1. Position of the points for Lemma 2.2.

Definition 2.1. *With the above notation, we let $F \subseteq \mathbb{R}^2$ be the bounded set whose boundary is $\partial F = \partial E \setminus \widehat{xy} \cup xy$.*

It is to be noticed that the above definition makes sense: indeed, by [10, Theorem 1.1] we already know that E is bounded, and by construction the segment xy cannot intersect any point of $\partial E \setminus \widehat{xy}$. In particular, all the connected components of E whose boundary is not γ belong also to F; instead, the connected component of E with γ as boundary has been slightly changed near z. Recalling that f is real-valued and strictly positive, as well as α-Hölder, we can find a constant M such that

$$\frac{1}{M} \leq f(p) \leq M \,, \qquad\qquad |f(p) - f(q)| \leq M|p-q|^\alpha \tag{2.1}$$

for every p, q in some neighborhood of E, containing all the points that we are going to use in our argument: this is not a problem, since all our arguments will be local. Let us now give the first easy estimates about the above quantities.

Lemma 2.2. *With the above notation, one has*

$$l \leq C\rho^{\frac{2}{2-\alpha}} \,, \qquad\qquad \delta \leq C\rho^{\frac{\alpha}{4-2\alpha}} \,, \tag{2.2}$$

and moreover

$$\ell_f(\widehat{xy}) - \ell_f(xy) \ll \ell(xy) \,. \tag{2.3}$$

Proof. First of all, let us consider a point $p \in E\Delta F$ in the symmetric difference between E and F; by construction, either p belongs to the ball $B_\rho(z)$, or it has a distance at most $l/2$ from that ball. As a consequence, p has distance at most $\rho + l/2$ from z, and this gives

$$|V_f(E) - V_f(F)| \leq V_f(E\Delta F) \leq M\Big(\pi(\rho + l/2)^2\Big) \leq 2M\pi(\rho^2 + l^2) \,. \tag{2.4}$$

Let us now apply Theorem 1.2 to the set E, with a ball B intersecting ∂E far away from the point z. We get a constant $\bar\varepsilon$ and of course, up to have taken ρ small enough, we can assume that $|\varepsilon|\leq\bar\varepsilon$, being $\varepsilon=V_f(E)-V_f(F)$. Then, Theorem 1.2 provides us with a set $\widetilde E$ satisfying (1.1); as a consequence, if we define $\widetilde F=F\setminus B\cup(B\cap\widetilde E)$, we get $V_f(\widetilde F)=V_f(F)+V_f(\widetilde E)-V_f(E)=V_f(E)$, and then, since E is isoperimetric, by (2.4) we get

$$P_f(E)\leq P_f(\widetilde F)=P_f(F)+P_f(\widetilde E)-P_f(E)\leq P_f(F)+C\big(\rho^2+l^2\big)^{\frac{1}{2-\alpha}}\,,$$

which implies

$$\ell_f(\widehat{xy})-\ell_f(xy)=P_f(E)-P_f(F)\leq C\big(\rho^2+l^2\big)^{\frac{1}{2-\alpha}}\,. \tag{2.5}$$

Let us now evaluate the term $\ell_f(\widehat{xy})-\ell_f(xy)$: if we call $f_{\min}$ and $f_{\max}$ the minimum and the maximum of f inside $B_z(\rho)$, by (2.1) we have

$$f_{\max}\leq f_{\min}+2M\rho^\alpha\,,\qquad\qquad f_{\min}\geq\frac{1}{M}\,,$$

and then

$$\begin{aligned}\ell_f(\widehat{xy})-\ell_f(xy)&=\ell_f(\widehat{\bar x\,\bar y})+\ell_f(\widehat{x\bar x})+\ell_f(\widehat{y\bar y})-\ell_f(xy)\geq 2\rho f_{\min}+\frac{l}{M}-2\rho\cos(\delta)f_{\max}\\&\geq 2\rho f_{\min}+\frac{l}{M}-2\rho\cos(\delta)\big(f_{\min}+2M\rho^\alpha\big)\geq 2\,\frac{1-\cos\delta}{M}\,\rho+\frac{l}{M}-4M\rho^{\alpha+1}\,.\end{aligned}$$

Inserting this estimate in (2.5), we get then

$$2\,\frac{1-\cos\delta}{M}\,\rho+\frac{l}{M}\leq C\big(\rho^2+l^2\big)^{\frac{1}{2-\alpha}}+4M\rho^{\alpha+1}\,.$$

Since

$$\frac{2}{2-\alpha}>1\,,\qquad\qquad \alpha+1\geq\frac{2}{2-\alpha}\,,$$

we immediately derive first that δ is very small, so that $1-\cos\delta\geq\delta^2/3$, and then

$$\delta^2\rho+l\leq C\rho^{\frac{2}{2-\alpha}}\,.$$

This gives the validity of both the inequalities in (2.2). Finally, (2.5) together with (2.2) and the fact that, since $\delta\ll 1$, one has $\ell(xy)\approx 2\rho$, gives (2.3). ☐

Corollary 2.3. *For any two points $a,\,b\in\gamma$ sufficiently close to each other, one always has*

$$\ell_f(\widehat{ab})-\ell_f(ab)\ll\ell_f(ab)\,,\qquad\qquad \ell(\widehat{ab})-\ell(ab)\ll\ell(ab)\,. \tag{2.6}$$

Proof. Let $z\in\widehat{ab}$ be a point such that $\rho=\ell(az)=\ell(zb)$, and let us call x, $\bar x$, y and $\bar y$ as before. Of course, $a\in\widehat{x\bar x}$ and $b\in\widehat{\bar y y}$, so that $\ell(a\bar x)+\ell(b\bar y)\leq l$. Since by (2.2) we have $l\ll\rho$, we get $a\hat z\bar x\ll 1$ and $b\hat z\bar y\ll 1$, as well as $\ell_f(ab)\approx\ell_f(xy)$. Hence, (2.3) gives us

$$\ell_f(\widehat{ab})-\ell_f(ab)\leq\ell_f(\widehat{xy})-\ell_f(xy)+\ell_f(xy)-\ell_f(ab)\ll\ell(ab)\,,$$

and then the first estimate in (2.6) is established. The second one immediately follows, just thanks to the continuity of f. ☐

An argument similar to the one proving Lemma 2.2 gives then the following estimate.

Lemma 2.4. *Given any two sufficiently close points r, $s \in \gamma$, one has*

$$\ell_f(\widehat{rs}) - \ell_f(rs) \geq -12M^5\,\ell(rs)^{\frac{2+\alpha}{2-\alpha}}\,. \tag{2.7}$$

Proof. Let us call t the maximal distance between points of the arc $\widehat{rs}$ and the segment rs, and let us denote $2d = \ell(rs)$ for brevity. Of course, concerning the Euclidean distances, one has

$$\ell(\widehat{rs}) \geq 2\sqrt{d^2+t^2}\,. \tag{2.8}$$

On the other hand, let us call π the projection of $\mathbb{R}^2$ on the line containing the segment rs, so that for every $a \in \widehat{rs}$ one has $|a-\pi(a)| \leq t$ by definition; moreover, define the density $g:\mathbb{R}^2\to\mathbb{R}^+$ as $g(a) = f(\pi(a))$, so that by the α-Hölder property (2.1) of f we get

$$f(a) \geq g(a) - Mt^\alpha \tag{2.9}$$

for every $a\in\widehat{rs}$. Since $\pi(a)=a$ for every $a\in rs$, by (2.8) and (2.1) we can then easily evaluate

$$\ell_g(\widehat{rs}) - \ell_f(rs) = \ell_g(\widehat{rs}) - \ell_g(rs) \geq \frac{2}{M}\left(\sqrt{d^2+t^2}-d\right).$$

On the other hand, by (2.9) and by (2.6) we have also

$$\ell_f(\widehat{rs}) - \ell_g(\widehat{rs}) \geq -Mt^\alpha\ell_f(\widehat{rs}) \geq -2Mt^\alpha\ell_f(rs) \geq -4M^2t^\alpha d\,.$$

Putting together the last two estimates, we obtain then

$$\ell_f(\widehat{rs}) - \ell_f(rs) = \ell_g(\widehat{rs}) - \ell_f(rs) + \ell_f(\widehat{rs}) - \ell_g(\widehat{rs}) \geq \frac{2}{M}\left(\sqrt{d^2+t^2}-d\right) - 4M^2t^\alpha d\,.$$

As a consequence, we can assume that $t \ll d$, since otherwise we readily get $\ell_f(\widehat{rs}) - \ell_f(rs) > 0$, and in this case of course (2.7) holds. Therefore, the last inequality can be rewritten as

$$\ell_f(\widehat{rs}) - \ell_f(rs) \geq \frac{2t^2}{3Md} - 4M^2t^\alpha d = \left(\frac{2t^2}{3Md^2} - 4M^2t^\alpha\right)d\,. \tag{2.10}$$

There are then two cases: if the term between parenthesis is positive, then again (2.7) clearly holds. If, instead, it is negative, this implies

$$t \leq 6M^3 d^{\frac{2}{2-\alpha}}\,,$$

and then (2.10) gives

$$\ell_f(\widehat{rs}) - \ell_f(rs) \geq -4M^2t^\alpha d \geq -24M^5 d^{\frac{2+\alpha}{2-\alpha}} \geq -12M^5\ell(rs)^{\frac{2+\alpha}{2-\alpha}}\,,$$

which is (2.7). □

We can now show a more refined estimate for $\ell_f(\widehat{pq}) - \ell_f(pq)$, which takes into account the maximal angle of deviation of the curve $\widehat{pq}$ with respect to the segment pq. Let us be more precise: for every $w\in\widehat{pq}$ we define $H = H(w)$ the projection of w on the line containing pq. Moreover, we call $\theta = \theta(w)$ the angle $w\widehat{q}p$ if H is closer to p than to q, and $\theta = w\widehat{p}q$ otherwise, and we let $\bar\theta$ be the maximum among all the angles $\theta(w)$ for $w\in\widehat{pq}$: Figure 2 depicts the situation.

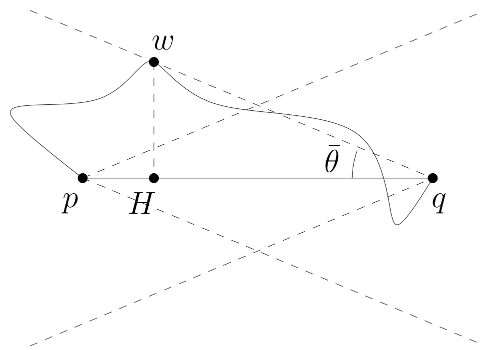

FIGURE 2. Definition of $\bar\theta$.

Lemma 2.5. *With the above notation, and calling $\rho = \ell(pq)$, there is $C = C(\alpha, M)$ for which*

$$\begin{aligned}
&\text{either}\quad \bar\theta \leq C\rho^{\frac{\alpha}{2-\alpha}}, &&\text{and then}\quad \ell_f(\widehat{pq}) - \ell_f(pq) \geq -C\rho^{\frac{2+\alpha}{2-\alpha}}\,,\\
&\text{or}\quad \bar\theta \geq C\rho^{\frac{\alpha}{2-\alpha}}, &&\text{and then}\quad \ell_f(\widehat{pq}) - \ell_f(pq) \geq \frac{\rho}{12M}\,\bar\theta^2\,.
\end{aligned} \tag{2.11}$$

Proof. Let us fix a point $w \in \widehat{pq}$ such that $\theta(w) = \bar\theta$, and let assume without loss of generality that $\bar\theta = w\widehat{q}p$, as in Figure 2. We can then apply Lemma 2.4, first with $rs = pw$, and then with $rs = wq$, to get, also keeping in mind (2.6), that

$$\begin{aligned}
\ell_f(\widehat{pq}) &= \ell_f(\widehat{pw}) + \ell_f(\widehat{wq}) \geq \ell_f(pw) + \ell_f(wq) - 12M^5\Big(\ell(pw)^{\frac{2+\alpha}{2-\alpha}} + \ell(wq)^{\frac{2+\alpha}{2-\alpha}}\Big)\\
&\geq \ell_f(pw) + \ell_f(wq) - 24M^5\rho^{\frac{2+\alpha}{2-\alpha}}\,.
\end{aligned} \tag{2.12}$$

We have now to compare the lengths of the segments pw and qw with those of the segments pH and qH. We can again argue defining the density g as $g(a) = f(\pi(a))$, being π the projection on the line containing the segment pq: then

$$\begin{aligned}
\ell_f(pw) - \ell_f(pH) &= \ell_g(pw) - \ell_g(pH) + \ell_f(pw) - \ell_g(pw)\\
&\geq \frac{1}{M}\Big(\ell(pw) - \ell(pH)\Big) - M\ell_f(pw)\ell(wH)^\alpha\,,
\end{aligned}$$

and the analogous estimate of course works for $\ell_f(wq) - \ell_f(wH)$. Putting them together, and recalling that $\ell_f(pH) + \ell_f(qH) \geq \ell_f(pq)$, with strict inequality if H is outside pq, we obtain

$$\begin{aligned}
\ell_f(pw) + \ell_f(wq) - \ell_f(pq) &\geq \frac{1}{M}\Big(\ell(pw) + \ell(wq) - \ell(pq)\Big) - M\Big(\ell_f(pw) + \ell_f(wq)\Big)\ell(wH)^\alpha\\
&\geq \frac{\rho}{6M}\bar\theta^2 - 2M^2\rho^{1+\alpha}\bar\theta^\alpha\,,
\end{aligned}$$

where we have again used that $\bar\theta \ll 1$, which comes as usual by (2.6). Putting this estimate together with (2.12), we get

$$\ell_f(\widehat{pq}) - \ell_f(pq) \geq \frac{\rho}{6M}\,\bar\theta^2 - 2M^2\rho^{1+\alpha}\bar\theta^\alpha - 24M^5\rho^{\frac{2+\alpha}{2-\alpha}}\,. \tag{2.13}$$

Now, notice that

$$\rho^{1+\alpha}\bar\theta^\alpha \geq \rho^{\frac{2+\alpha}{2-\alpha}} \qquad\Longleftrightarrow\qquad \bar\theta \geq \rho^{\frac{\alpha}{2-\alpha}} \qquad\Longleftrightarrow\qquad \rho\bar\theta^2 \geq \rho^{1+\alpha}\bar\theta^\alpha\,.$$

As a consequence, (2.13) implies the validity of both the cases in (2.11), up to have chosen a sufficiently large constant $C = C(M, \alpha)$. $\square$

We are now ready to find a first result about the behaviour of the direction of the chords connecting points of the curve γ. Indeed, putting together Lemma 2.5 and Lemma 2.2, we get the next estimate.

Lemma 2.6. *Let a, z be two points in γ sufficiently close to each other, call $\rho = \ell(az)$, and let $w \in \widehat{az}$ be a point closer to a than to z. Then, there is $C = C(M, \alpha)$ such that*

$$w\widehat{z}a \leq C\rho^{\frac{\alpha}{3-2\alpha}}\,.$$

Proof. Having a point $z \in \gamma$ and some $\rho > 0$ very small, we can define the points x, $\bar{x}$, y, $\bar{y}$ as for Lemma 2.2, and by symmetry we can think $a \in \widehat{x\bar{x}}$. We apply now twice Lemma 2.5, once with $p_1 = x$ and $q_1 = z$, and once with $p_2 = y$ and $q_2 = z$. We have then the validity of (2.11) for the two angles $\bar{\theta}_1$ and $\bar{\theta}_2$; we claim that

$$\bar{\theta} := \max\left\{\bar{\theta}_1,\, \bar{\theta}_2\right\} \leq C\rho^{\frac{\alpha}{3-2\alpha}}\,. \tag{2.14}$$

Let us first see that this estimate implies the thesis. If the point w is closer to x than to z, then by definition and by (2.14)

$$w\widehat{z}x = \theta_1(w) \leq \bar{\theta}_1 \leq C\rho^{\frac{\alpha}{3-2\alpha}}\,.$$

As a consequence, by (2.2) we have

$$w\widehat{z}a \leq w\widehat{z}x + x\widehat{z}a \leq C\rho^{\frac{\alpha}{3-2\alpha}} + 2\,\frac{l}{\rho} \leq C\rho^{\frac{\alpha}{3-2\alpha}} + C\rho^{\frac{\alpha}{2-\alpha}} \leq C\rho^{\frac{\alpha}{3-2\alpha}}\,, \tag{2.15}$$

and the thesis is obtained. Suppose, instead, that w is closer to z than to x; since by assumption it is anyhow closer to a than to z, and $\ell(ax) \leq l \ll \ell(wz)$ by (2.2), again using (2.14) we have

$$w\widehat{z}x \leq 2w\widehat{x}z = 2\theta_1(w) \leq C\rho^{\frac{\alpha}{3-2\alpha}}\,,$$

then the very same argument as in (2.15) shows again the thesis. Summarizing, we have proved that (2.14) implies the thesis, and hence to conclude we only have to show the validity of (2.14).

We argue as in Lemma 2.2, defining the competitor set F which has $\partial F = \partial E \setminus \widehat{xy} \cup xz \cup zy$ as boundary. Notice that this is very similar to what we did in Definition 2.1, the only difference being that we are putting the two segments xz and zy instead of the segment xy. The very same argument that we presented after Definition 2.1 still ensures that the set F is well defined. As in Lemma 2.2, then, we have now to evaluate $|V_f(F) - V_f(E)|$ and $P_f(F) - P_f(E)$. Concerning the first quantity, we can get an estimate which is much better than (2.4), thanks to the definition of $\bar{\theta}_1$ and $\bar{\theta}_2$.

Let us be more precise. The curve $\widehat{xz}$ is composed by two pieces; concerning the first part, $\widehat{x\bar{x}}$, by definition this remains within a distance of at most $\ell(\widehat{x\bar{x}})$ from x. The second part, $\widehat{\bar{x}z}$, is contained in the shaded region of Figure 3, which is the intersection between the ball $B_\rho(z)$ and the region of the points w such that $\min\{w\widehat{x}z,\, w\widehat{z}x\} \leq \bar{\theta}_1$. Repeating the same argument with $\bar{\theta}_2$, y and $\bar{y}$, and recalling that $l = \ell(\widehat{x\bar{x}}) + \ell(\widehat{y\bar{y}})$, we have

$$|V_f(E) - V_f(F)| \leq 9M\rho^2\big(\bar{\theta}_1 + \bar{\theta}_2\big) + \pi Ml^2 \leq 18M\rho^2\bar{\theta} + \pi Ml^2\,.$$

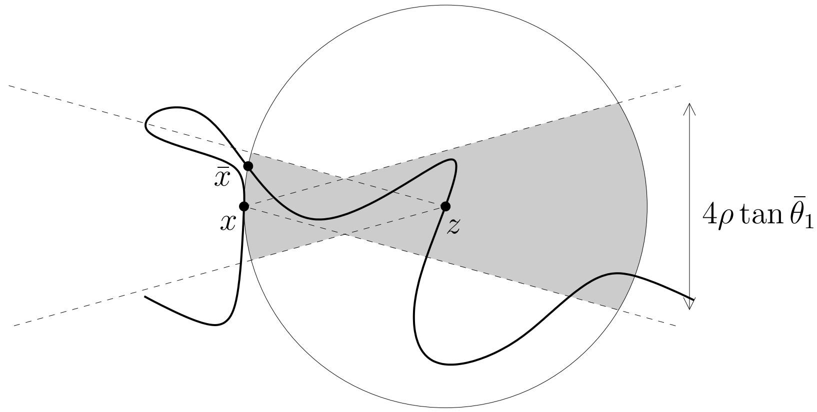

FIGURE 3. Constraint on the position of the curve $\widehat{xz}$.

Let us now observe that by (2.2) one has $l^2 \leq C\rho^{\frac{4}{2-\alpha}}$, and then

$$l^2 \leq \rho^2\bar\theta \quad \Longleftarrow \quad C\rho^{\frac{4}{2-\alpha}} \leq \rho^2\bar\theta \quad \Longleftrightarrow \quad \bar\theta \geq C\rho^{\frac{2\alpha}{2-\alpha}}\,.$$

There are then two possibilities: either $\bar\theta \leq C\rho^{\frac{2\alpha}{2-\alpha}}$, so we already have the validity of (2.14) and the proof is concluded, or $\bar\theta \geq C\rho^{\frac{2\alpha}{2-\alpha}}$, and then the last two estimates imply

$$|V_f(E) - V_f(F)| \leq 22M\rho^2\bar\theta\,.$$

Therefore, using Theorem 1.2 exactly as in the proof of Lemma 2.2, we find a set $\widetilde F$ having the same volume as E (and then more perimeter) satisfying

$$P_f(E) \leq P_f(\widetilde F) \leq P_f(F) + C\Big(22M\rho^2\bar\theta\Big)^{\frac{1}{2-\alpha}} = P_f(F) + C\rho^{\frac{2}{2-\alpha}}\bar\theta^{\frac{1}{2-\alpha}}\,,$$

from which we directly get

$$\ell_f(\widehat{xy}) - \ell_f(xz) - \ell_f(zy) = P_f(E) - P_f(F) \leq C\rho^{\frac{2}{2-\alpha}}\bar\theta^{\frac{1}{2-\alpha}}\,. \tag{2.16}$$

We now use the fact that (2.11) is valid both with $\bar\theta_1$ and with $\bar\theta_2$, as pointed out before. One has to distinguish three possible cases.

Since $\rho^{\frac{\alpha}{2-\alpha}} \leq \rho^{\frac{\alpha}{3-2\alpha}}$, if both $\bar\theta_1$ and $\bar\theta_2$ are smaller than $C\rho^{\frac{\alpha}{2-\alpha}}$ then so is $\bar\theta$, so (2.14) is true and there is nothing more to prove.

Suppose now that both $\bar\theta_1$ and $\bar\theta_2$ are bigger than $C\rho^{\frac{\alpha}{2-\alpha}}$. In this case, (2.11) gives

$$\ell_f(\widehat{xy}) - \ell_f(xz) - \ell_f(zy) = \Big(\ell_f(\widehat{xz}) - \ell_f(xz)\Big) + \Big(\ell_f(\widehat{zy}) - \ell_f(zy)\Big) \geq \frac{\rho}{12M}\left(\bar\theta_1^2 + \bar\theta_2^2\right) \geq \frac{\rho}{12M}\,\bar\theta^2\,,$$

which together with (2.16) gives

$$\rho\bar\theta^2 \leq C\rho^{\frac{2}{2-\alpha}}\bar\theta^{\frac{1}{2-\alpha}}\,,$$

which is equivalent to (2.14), and then also in this case the proof is completed.

Finally, we have to consider what happens when only one between $\bar\theta_1$ and $\bar\theta_2$ is bigger than $C\rho^{\frac{\alpha}{2-\alpha}}$; just to fix the ideas, we can suppose that $\bar\theta_1 \geq C\rho^{\frac{\alpha}{2-\alpha}} \geq \bar\theta_2$, hence in particular $\bar\theta = \bar\theta_1$. Applying then (2.11), this time we find

$$\ell_f(\widehat{xy}) - \ell_f(xz) - \ell_f(zy) = \Big(\ell_f(\widehat{xz}) - \ell_f(xz)\Big) + \Big(\ell_f(\widehat{zy}) - \ell_f(zy)\Big) \geq \frac{\rho}{12M}\,\bar\theta^2 - C\rho^{\frac{2+\alpha}{2-\alpha}} \geq \frac{\rho}{24M}\,\bar\theta^2\,,$$

where the last inequality is true precisely because $\bar\theta \geq C\rho^{\frac{\alpha}{2-\alpha}}$. Exactly as before, putting this estimate together with (2.16) implies (2.14), and then also in the last possible case we obtained the proof. □

The last lemma is exactly what we needed to obtain the proof of our main Theorem A.

Proof (of Theorem A). Let E be an isoperimetric set for the $\mathrm{C}^{0,\alpha}$ density f. To show that ∂E is of class $\mathrm{C}^{1,\frac{\alpha}{3-2\alpha}}$, let us select two generic points z, $a\in\partial E$ such that $\rho=\ell(za)\ll 1$. We want to show that

$$w\widehat{z}a\leq C\rho^{\frac{\alpha}{3-2\alpha}}\qquad \forall w\in\widehat{az}\,, \tag{2.17}$$

as this readily imply the thesis. Indeed, suppose that (2.17) has been established, and call $\nu\in\mathbb{S}^1$ the direction of the segment az; since we already know that ∂E is of class C^1 by Theorem 1.3, considering points $w\in\widehat{az}$ which converge to z we deduce by (2.17) that $|\nu-\nu_z|\leq C\rho^{\frac{\alpha}{3-2\alpha}}$, where $\nu_z\in\mathbb{S}^1$ is the tangent vector of ∂E at z. Since the situation of a and of z is perfectly symmetric, the same argument also shows that $|\nu-\nu_a|\leq C\rho^{\frac{\alpha}{3-2\alpha}}$, thus by triangual inequality we have found

$$|\nu_a-\nu_z|\leq C\rho^{\frac{\alpha}{3-2\alpha}}\,.$$

Since a and z are two generic points having distance ρ, and since C only depends on M and on α, this estimate shows that ∂E is of class $\mathrm{C}^{\frac{\alpha}{3-2\alpha}}$; therefore, the proof will be concluded once we show (2.17). Notice that (2.17) simply says that the estimate of Lemma 2.6 holds also without asking to the point w to be closer to a than to z.

To do so, let us recall that by (2.6) it is $\ell(\widehat{az})\leq 2\rho$, let us write $a_0=a$, and let us define recursively the sequence a_j by letting $a_{j+1}\in\widehat{a_jz}$ be the point such that

$$\ell(\widehat{a_{j+1}z})=\frac{2}{3}\,\ell(\widehat{a_jz})\,.$$

Observe that a_j is a sequence inside the curve $\widehat{az}$, which converges to z, and moreover for every $j\in\mathbb{N}$ one has

$$\ell(a_jz)\leq\ell(\widehat{a_jz})=\left(\frac{2}{3}\right)^j\ell(\widehat{az})\leq 2\left(\frac{2}{3}\right)^j\rho\,. \tag{2.18}$$

Let us now take a point $w\in\widehat{a_ja_{j+1}}$; again recalling (2.6), by the definition of the points a_j it is obvious that w is closer to a_j than to z; as a consequence, Lemma 2.6 applied to a_j and z ensures that

$$w\widehat{z}a_j\leq C\ell(a_jz)^{\frac{\alpha}{3-2\alpha}}\leq C\,\kappa^j\rho^{\frac{\alpha}{3-2\alpha}}\qquad\forall\, w\in\widehat{a_ja_{j+1}}\,,$$

where we have also used (2.18), and where $\kappa=(2/3)^{\frac{\alpha}{3-2\alpha}}<1$. Keeping in mind the obvious fact that $a_{j+1}\in\widehat{a_ja_{j+1}}$ for every j, and then the above estimate is valid in particular for the point a_{j+1}, we deduce that for the generic $w\in\widehat{a_ja_{j+1}}$ it is

$$w\widehat{z}a=w\widehat{z}a_0\leq w\widehat{z}a_j+\sum_{i=0}^{j-1}a_{i+1}\widehat{z}a_i\leq C\rho^{\frac{\alpha}{3-2\alpha}}\sum_{i=0}^{j}\kappa^i\leq C\rho^{\frac{\alpha}{3-2\alpha}}\,,$$

where the last inequality is true because $\kappa=\kappa(\alpha)<1$. We have then established (2.17), and then the proof is concluded. □

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